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# A coloring problem on the $n$-cube

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**Abstract**

In this paper, we consider a coloring problem on the $n$-cube that arises in the study of scalability of optical networks. Let $\chi_{\bar{k}}(n)$ be the minimum number of colors needed to color the vertices of the $n$-cube so that every two vertices with Hamming distance $\leqslant k$ have different colors. We show that for $k=3$, $2n \leqslant \chi_{\bar{3}}(n) \leqslant 2^{\lceil \log_2 n \rceil + 1}$. We also provide upper and lower bounds on $\chi_{\bar{k}}(n)$ for general $k$. 

## 1. Introduction

An $n$-cube is a graph with vertices $(x_1, x_2, \ldots, x_n)$ for $x_1, x_2, \ldots, x_n \in \{0, 1\}$, and edges $(x, y)$ for vertices $x$ and $y$ with Hamming distance one, where the Hamming distance is defined by $d(x, y) = |\{i \mid x_i \neq y_i\}|$. Given a positive integer $k$, how many colors does an $n$-cube need to color its vertices so that any two vertices with distance exactly $k$ have different colors? Or so that any two vertices within distance $k$ have different colors? These two problems were proposed in the study of scalability of optical networks [3,5]. The second one is equivalent to finding the chromatic number of the $k$th power of the $n$-cube (The $k$th power $G^k$ of a graph $G$ has the same vertex-set as $G$, with an edge between two vertices whenever they are within distance $k$ in $G$). Let $\chi_k(n)$, respectively $\chi_{\bar{k}}(n)$, denote the minimum number of colors in a vertex-coloring of

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the $n$-cube such that any two vertices with distance exactly $k$, respectively at most $k$, have different colors. Clearly, $\chi_k(n) \leqslant \chi_{\bar{k}}(n)$ for each $n$, and $\chi_k(n) = 2$ when $k$ is odd. Wan [5] showed that

$$n \leqslant \chi_2(n) \leqslant 2^{\lceil \log_2 n \rceil}, \tag{1}$$

$$n + 1 \leqslant \chi_{\bar{2}}(n) \leqslant 2^{\lceil \log_2 (n+1) \rceil}. \tag{2}$$

He also conjectured that $\chi_2(n) = 2^{\lceil \log_2 n \rceil}$ and $\chi_{\bar{2}}(n) = 2^{\lceil \log_2 (n+1) \rceil}$. In this paper, we show that

$$2n \leqslant \chi_{\bar{3}}(n) \leqslant 2^{\lceil \log_2 n \rceil + 1}. \tag{3}$$

Note that if $n$ is a power of 2 then the upper and lower bounds in (1) and (3) coincide, so that $\chi_2(n) = n$ and $\chi_{\bar{3}}(n) = 2n$. If $n + 1$ is a power of 2 then the upper and lower bounds in (2) coincide, so that $\chi_{\bar{2}}(n) = n + 1$. We also find some general lower and upper bounds for $\chi_{\bar{k}}(n)$ and $\chi_k(n)$.

## 2. Main results

An *m-dimensional binary representation* of an integer $i$ is an $m$-dimensional vector $(x_1, x_2, \ldots, x_m)$ such that $x_1 \cdot 2^{m-1} + x_2 \cdot 2^{m-2} + \cdots + x_m = i$. Clearly, the nonnegative integer $i$ has an $m$-dimensional binary representation if and only if $\lceil \log_2 (i+1) \rceil \leqslant m$.

**Theorem 1.** $2n \leqslant \chi_{\bar{3}}(n) \leqslant 2^{\lceil \log n \rceil + 1}$.

**Proof.** To establish the lower bound, consider two vertices $(0, 0, \ldots, 0)$ and $(1, 0, \ldots, 0)$ and their adjacent vertices. These $2n$ vertices have the property that the distance between any two of them is at most three. Therefore, they should have distinct colors. That is, $2n \leqslant \chi_{\bar{3}}(n)$.

To establish the upper bound, let $m = (\lceil \log_2 n \rceil + 1)$ and let $M(n)$ be the $n$-by-$m$ matrix whose $(i+1)$th row $r_i$ $(0 \leqslant i \leqslant n-1)$ has the form $(b_i, n_i)$, where $b_i$ is the $(m-1)$-dimensional binary representation of integer $i$ and

$$n_i = \begin{cases} 1 & \text{if } b_i \text{ contains an even number of 1's,} \\ 0 & \text{otherwise.} \end{cases}.$$

For example,

$$M(5) = \begin{pmatrix} 0 & 0 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 1 \\ 1 & 0 & 0 & \vdots & 0 \end{pmatrix}.$$

Now, we denote colors by $m$-dimensional 0–1 vectors and assume that each vertex $x=(x_1,x_2,\dots,x_n)$ is in color $xM(n)$ where all arithmetic operations are performed in the ring $Z_2$. Note that we use $2^m$ colors in total.

If $x$ and $y$ are vertices with distance $d(x,y)=d\geqslant 1$, then $xM(n)-yM(n)$ (evaluated modulo 2) is the mod-2 sum of the $d$ rows of $M(n)$ corresponding to the coordinate positions in which $x$ and $y$ differ. If $d\leqslant 3$, this is nozero, since no two rows of $M(n)$ are equal, and any odd number of rows of $M(n)$ contain an odd number of 1's in total. Therefore, if $d(x,y)\leqslant 3$, then $xM(n)\neq yM(n)$, and so vertices $x$ and $y$ have different colors. Hence $\chi_{\bar{3}}(n)\leqslant 2^m$. □

**Lemma 1.** *If* $x_1,x_2,\dots,x_h,y_1,y_2,\dots,y_h$ *are real numbers such that* $x_1^u+x_2^u+\cdots+x_h^u=y_1^u+y_2^u+\cdots+y_h^u$ *for* $u=1,2,\dots,h$, *then* $\{x_1,x_2,\dots,x_h\}=\{y_1,y_2,\dots,y_h\}$.

**Proof.** Let $\sigma_u(x)=\sum_{1\leqslant i_1<i_2<\cdots<i_u\leqslant h}x_{i_1}x_{i_2}\cdots x_{i_u}$. Then $\sigma_u(x)$ is a function of $x_1^k+x_2^k+\cdots+x_h^k$ for $k=1,2,\dots,u$. Thus, $\sigma_u(x)=\sigma_u(y)$ $(=\sigma_u$, say) for $u=1,2,\dots,h$. This means that both $\{x_1,x_2,\dots,x_h\}$ and $\{y_1,y_2,\dots,y_h\}$ are the set of roots for equation $t^h-\sigma_1t^{h-1}+\cdots+(-1)^h\sigma_h=0$. Thus, $\{x_1,x_2,\dots,x_h\}=\{y_1,y_2,\dots,y_h\}$. □

**Theorem 2.**

$$\left(\!\!\binom{n}{k/2}\!\!\right)\leqslant\chi_{\bar{k}}(n)\leqslant(k+1)\left(\frac{k+2}{2}\right)^{(k(k+2))/8\lceil\log_2 n\rceil}\qquad \textit{for even } k,$$

$$2\left(\!\!\binom{n-1}{(k-1)/2}\!\!\right)\leqslant\chi_{\bar{k}}(n)\leqslant(k+1)\left(\frac{k+1}{2}\right)^{(k-1)(k+1)/8\lceil\log_2 n\rceil}\qquad \textit{for odd } k,$$

*where* $\left(\!\binom{n}{k}\!\right)=\sum_{i=0}^{k}\binom{n}{i}$.

**Proof.** To establish the lower bound for even $k$, we consider all vertices within distance $k/2$ from $(0,0,\dots,0)$. The total number of these vertices is $\left(\!\binom{n}{k/2}\!\right)$. The distance between any two of them is at most $k$. Thus, these vertices should have distinct colors.

To establish the lower bound for odd $k$, we consider all vertices within distance $(k-1)/2$ of either vertex $(0,0,\dots,0)$ or $(1,0,\dots,0)$. The total number of these vertices is $2\left(\!\binom{n-1}{(k-1)/2}\!\right)$. The distance between any two of them is at most $k$. Thus, these vertices should have distinct colors.

To establish the upper bound for even $k$, note that

$$\lceil\log_2((n-1)^u+1)\rceil\leqslant\lceil\log_2(n^u)\rceil\leqslant u\lceil\log_2 n\rceil$$

and let

$$m=1+\sum_{u=1}^{k/2}\lceil\log_2((n-1)^u+1\rceil\leqslant 1+\frac{k(k+2)}{8}\lceil\log_2 n\rceil.$$

Let $M_k(n)$ be the $n$-by-$m$ matrix $M_k(n)$ whose $(i+1)$th row $r_i$ has the form $r_i=(1,b_i^1,b_{i^2}^2,\dots,b_{i^{k/2}}^{k/2})$ where $b_j^u$ is a $\lceil\log_2((n-1)^u+1)\rceil$-dimensional binary representation of nonnegative integer $j$. For example,

$$M_4(6)=\begin{pmatrix} 1 & \vdots & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & 0 & 0 \\ 1 & \vdots & 0 & 0 & 1 & \vdots & 0 & 0 & 0 & 0 & 1 \\ 1 & \vdots & 0 & 1 & 0 & \vdots & 0 & 0 & 1 & 0 & 0 \\ 1 & \vdots & 0 & 1 & 1 & \vdots & 0 & 1 & 0 & 0 & 1 \\ 1 & \vdots & 1 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 & 0 \\ 1 & \vdots & 1 & 0 & 1 & \vdots & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Denote colors by $m$-dimensional vectors $(c_1,c_2,\ldots,c_m)$ where $c_1 \in Z_{k+1},\ c_2,c_3,\ldots,c_m \in Z_{k/2+1}$. Thus, the total number of colors is

$$(k+1)(k/2+1)^{m-1} \leqslant (k+1)\left(\frac{k+2}{2}\right)^{(k(k+2))/8\lceil \log_2 n\rceil}.$$

Assign the color $xM_k(n)$ to vertex $x$ where $xM_k(n)$ is calculated as follows: The first column is computed with operations in the ring $Z_{k+1}$, i.e., do addition modulo $k+1$. The other columns are computed with operations in the ring $Z_{k/2+1}$. For example,

$$(0,1,0,1,1,1)M_4(6)=(4,2,1,0,2,2,0,0,0).$$

We now show that for any two vertices $x$ and $y$ within distance $k$, $xM_k(n) \neq yM_k(n)$. For a contradiction, suppose that $xM_k(n)=yM_k(n)$. From the first column, we see that the vectors $x$ and $y$ have the same number of 1's. So, for some $h \leqslant k/2$ there are $h$ coordinate positions where $x$ has 1's and $y$ has 0's, say $i_1,\ldots,i_h$, and $h$ where $y$ has 1's and $x$ has 0's, say $j_1,\ldots,j_h$. Then

$$r_{i_1}+r_{i_2}+\cdots+r_{i_h}=r_{j_1}+r_{j_2}+\cdots+r_{j_h},$$

so that

$$b^u_{(i_1)^u}+b^u_{(i_2)^u}+\cdots+b^u_{(i_h)^u} \equiv b^u_{(j_1)^u}+b^u_{(j_2)^u}+\cdots+b^u_{(j_h)^u} \pmod{(k/2+1)} \tag{4}$$

for $u=1,2,\ldots,k/2$. Since $h \leqslant k/2$, the two sides of (4) are not only congruent but equal, so that

$$i_1^u+i_2^u+\cdots+i_h^u=j_1^u+j_2^u+\cdots+j_h^u$$

for $u=1,2,\ldots,k/2$. By Lemma 1, we can conclude that $\{i_1,i_2,\ldots,i_h\}=\{j_1,j_2,\ldots,j_h\}$, a contradiction.

Similarly, we can establish the upper bound for $\chi_{\bar{k}}(n)$ for odd $k$. $\square$

**Corollary 1.** *For odd* $k$, $\chi_k(n)=2$. *For even* $k$, $\lfloor 2n/k\rfloor \leqslant \chi_k(n) \leqslant (k+1)\left(\frac{k+2}{2}\right)^{(k(k+2))/8\lceil \log_2 n\rceil}$.

**Proof.** Note that the $n$-cube is 2-colorable with every two vertices at odd distance having different colors. Therefore $\chi_k(n)=2$ for odd $k$. For even $k$, the upper bound follows from Theorem 2 since $\chi_k(n) \leqslant \chi_{\bar{k}}(n)$; the lower bound follows from considering vertices $(\underbrace{1,\ldots,1}_{k/2},0,\ldots,0)$, $(\underbrace{0,\ldots,0}_{k/2},\underbrace{1,\ldots,1}_{k/2},0,\ldots,0),\ldots$ . □

## 3. Discussion

The lower bounds presented in this paper are not tight. One way of improving the lower bound on $\chi_{\bar{k}}(n)$ is to consider the largest number of vertices that can have the same color, i.e., the largest number $A_2(n,d)$ of vertices in the $n$-cube whose minimum distance is at least $d$, where $d = k+1$. For example, it is easy to see that $A_2(5,4) = 2$ which means that each color can be used at most twice. Thus $\chi_{\bar{3}}(5) \geqslant 2^5/2$ and hence $\chi_{\bar{3}}(5) = 16 = 2^{\lceil \log_2 n \rceil + 1} > 2n = 10$ ($2n$ being the lower bound in Theorem 1). Since $\chi_{\bar{3}}(n)$ is a (weakly) increasing function of $n$ and $\chi_{\bar{3}}(8) = 16$ by Theorem 1, and since it is obvious that $\chi_{\bar{3}}(n) = 2^n = 2^{\lceil \log_2 n \rceil + 1}$ for $n \leqslant 3$, it follows that $\chi_{\bar{3}}(n) = 2^{\lceil \log_2 n \rceil + 1}$ if $n \leqslant 8$. We conjecture that $\chi_{\bar{3}}(n) = 2^{\lceil \log_2 n \rceil + 1}$ for all $n$. Note that this would follow if one could prove that $A_2(2^r+1,4) \leqslant 2^{2^r - r - 1}$ for every positive integer $r$. However, this inequality does not hold since $A_2(2^3+1,4) = A_2(2^3,3) = 20 > 16 = 2^{2^3-3-1}$ (see [4]). The value of $A_2(n,d)$ may provide only an improvement on the lower bound of $\chi_{\bar{k}}(n)$. Determining the value of $A_2(n,d)$ is an important problem in coding theory. It follows from Theorems 4.5.3, 4.5.4, and 4.5.7 in [4] that

$$\frac{2^{2^r}}{1+2^r+2^r(2^r-1)/2} \leqslant A_2(2^r+1,4) = A_2(2^r,3) \leqslant \frac{2^{2^r}}{1+2^r}.$$

This is also a possible way to improve the lower bound in the results of Wan [5]. The reader may refer to [1,2] for more information on the problem of finding $A_2(n,d)$.

There exists a large gap between the upper bound and the lower bound in Theorem 2. A small improvement on the upper bound can be obtained by using function $x^u - x^{u-1}$ instead of $x^u$. But, to close this gap, a new technique is required.

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